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Hereditary Semigroup Algebras[1]

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In "*Semigroup Algebras*," Okniński posed the following question: characterize semigroup algebras that are hereditary. In this paper we describe the (prime contracted) semigroup algebras $K[S]$ that are hereditary and Noetherian when S is either a Malcev nilpotent monoid, a cancellative monoid or a monoid extension of a finite non-null Rees matrix semigroup. Furthermore, for the class of monoids which have an ideal series with factors that are non-null Rees matrix semigroups, we obtain an upper bound for the global dimension of its contracted semigroup algebra. © 2000 Academic Press

1. INTRODUCTION

Hereditary and semihereditary rings have been the subject of considerable study. Many interesting examples of these rings arise as group rings or semigroup rings. Dicks has characterized the hereditary group rings [?, ?], earlier, Goursaud and Valette classified hereditary group rings of nilpotent groups [?]. Cheng and Wong [?] characterized the hereditary monoid

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rings that are also domains. In [?] and [?], Nico discussed the upper bound for homological dimensions of semigroup algebras $R[S]$ of a finite regular semigroup S over a commutative ring R. Recently, in [?], Kuzmanovich and Teply discovered bounds for homological dimensions of semigroup rings $R[S]$ of semigroups S which are monoids with a chain of ideals such that each factor semigroup is a finite non-null Rees matrix semigroup: the bounds are in terms of the dimension of the coefficient ring R and the structure of the semigroup S. In Sect. 2 we show that the bound can be sharpened. We do this by generalizing the method used by Nico in [?]. As an application we obtain a characterization of hereditary semigroup algebras of finite non-null Rees matrix semigroups when the coefficient ring is a field. In Sect. 3, we discuss semigroup algebras of semigroups that are nilpotent in the sense of Malcev ([?]). Theorem **??** characterizes when such algebras are hereditary Noetherian prime. In Sect. 4, a description is given of when a semigroup algebra of an arbitrary cancellative semigroup is hereditary Noetherian. If, moreover, S is a maximal order, then S is a finite p' or finite p'-by-infinite cyclic group, or S contains a finite p'-subgroup H and a non-periodic element x such that $S = \bigcup_{i\in N} Hx^i$, $xH = Hx$, and every central idempotent of $K[H]$ remains central in $K[S]$. Hence $K[S]$ is also a semiprime principal ideal ring by the results of [?]. Otherwise, S is a finite p'-by-D_∞ group.

Note that in this paper, hereditary (respectively, Noetherian) means both left and right hereditary (respectively, Noetherian). Recall the left global dimension of R equals the right global dimension when the ring R is Noetherian.

2. MONOID EXTENSIONS OF REES MATRIX SEMIGROUPS

In [?], Nico discussed the upper bound of the global dimension of $R[S]$ when the semigroup S is a finite regular monoid and R is a commutative ring with an identity. In [?], Kuzmanovich and Teply discussed the case when the semigroup S is a monoid with a chain of ideals $S = S_1 \supset S_2 \supset \cdots \supset S_l$ such that each factor semigroup S_i/S_{i+1} is a finite, non-null Rees matrix semigroup $\mathscr{M}^0(G_i; m_i, n_i; P_i)$. In particular, any finite semisimple semigroup S satisfies the above assumption. In this section, we exploit Nico's method to sharpen the upper bound discussed by Kuzmanovich and Teply in [?].

First, let us assume that S is a monoid with an ideal U such that U is isomorphic to a non-null Rees matrix semigroup $\mathscr{M}^0(G; n, m; P)$ with G a group and P the sandwich matrix. Let K be a ring with an identity. Let $I = K_0[U]$ and $\Lambda = K_0[S]$. Note that $K_0[U] = \mathscr{M}(K[G]; n, m; P)$, a Munn algebra. For notation and terminology we refer to [?] and [?].

We begin by extending Proposition 3.6 in [**?**].

LEMMA 2.1. *Under the above assumptions, the ideal $I = K_0[U]$ satisfies the following properties.*

(1) *There exist subsets A, B of U and an idempotent $e \in U$ such that $I = \bigoplus_{a\in A} aI = \bigoplus_{b\in B} Ib$. Moreover, $I = IeI = \Lambda e\Lambda$, $e\Lambda = eI$, and $\Lambda e = Ie$.*

(2) *For any $a \in A$ and $b \in B$, $ae = a$, $eb = b$ and thus $ba = ebae \in G \cup \{\theta\}$, where θ denotes the zero element.*

(3) *As a right Λ-module, $I = \bigoplus_{a\in A} aI$ is projective. Similarly, $I = \bigoplus_{b\in B} Ib$ is a left projective Λ-module.*

(4) *Ie is a left projective Λ-module. Considered as a right $K[G]$-module, $Ie \cong \bigoplus_{a\in A} aK[G]$ is free.*

(5) *Any nonzero element of I can be expressed as a sum of agb where $a \in A$, $b \in B$, and $g \in K[G]$.*

Proof. Without loss of generality, we can assume $P_{1,1} = 1$ (see also Remark 3.5 in [**?**]). Abusing notation, we identify $G \cup \{\theta\}$ with $\{(g, 1, 1) \mid g \in G \cup \{\theta\}\}$.

(1) Let $e = (1, 1, 1)$, that is e has a 1 in $(1, 1)$ entry and zero elsewhere. Then $e^2 = e \circ P \circ e = e$ is an idempotent and thus $e\Lambda = eI$ and $\Lambda e = Ie$. Here $\circ$ means the ordinary product of matrices. Clearly, $IeI \subseteq I$. Now we need to show that $I \subseteq IeI$. It is sufficient to show that, for an arbitrary element $a \in K[G]$, IeI contains a matrix that has a as its (i, j) entry and zero for its other entries. Indeed, let A_i be the matrix with 1 in the $(i, 1)$ entry and all other entries 0, and let C_j be the matrix with a in $(1, j)$ entry and all other entries 0. Then A_ieC_j has a as its (i, j) entry and zero for all its other entries. Hence $IeI = I$. Let $A = \{(1, i, 1) \mid 1 \le i \le n\}$. Choose A_i as before, clearly A_iI is the ith row of I, so $I = \bigoplus_{A_i\in A} A_iI$. Similarly, $I = \bigoplus_{B_j\in B} IB_i$ where $B = \{(1, 1, j) \mid 1 \le j \le m\}$ and B_j be the matrix with 1 in $(1, j)$ entry and all other entries 0.

(2) Follows from the proof of 1. Obviously, $A_ie = A_i \circ P \circ e = A_i$.

(3) Since $A_ie = A_i$, a direct computation shows that left multiplication by A_i yields a (right) Λ-isomorphism from eI to A_iI. Then $I = \bigoplus_{A_i\in A} A_iI \cong \bigoplus_{A_i\in A} eI$ is projective as a right Λ-module.

(4) As $Ie = \Lambda e$, it is clear that Ie is a left projective Λ-module. Hence, from the above,

$$\begin{aligned} Ie &= \bigoplus_{A_i\in A} A_iIe \\ &= \bigoplus_{A_i\in A} A_ieIe \\ &\cong \bigoplus_{A_i\in A} A_iK[G]. \end{aligned}$$

Since $A_i K[G] \cong K[G]$ as $K[G]$ modules, we obtain that Ie is a free right $K[G]$-module.

(5) Similar as in the proof of the first part (replace e by $ea \in K[G]$ and C_j by B_j). ∎

As in [?], for any left $\Lambda = K_0[S]$-module M, we define two modules M^* and M^{**} via the basic exact sequences:

$$0 \to \Lambda eM \to M \to M^* \to 0,$$

$$0 \to M^{**} \to \Lambda e \otimes_{K[G]} eM \xrightarrow{\delta} \Lambda eM \to 0.$$

Here $IeM = \Lambda eM$ is a submodule of M, eM is also a left $K[G]$-module; the map δ in the second sequence is given by $\beta \otimes m \longmapsto \beta m$.

Then we have the following lemma generalizing the lemma given in the completely 0-simple case discussed by Nico in [?].

LEMMA 2.2. *With M^* and M^{**} defined as above, $xM^* = xM^{**} = 0$ for all $x \in I$. Moreover, if the subalgebra I has a left identity, then $M^{**} = 0$ for every $K_0[S]$-module M.*

Proof. Because of Lemma **??**, $\Lambda eM = \Lambda e \Lambda M = IM$ and thus $M^* = M/IM$. Hence $xM^* = 0$ is obvious. By Lemma **??**.(4), any element $\alpha \in M^{**}$ can be written as $\alpha = \sum_{a \in A} a \otimes m_a$, where $m_a \in eM$ and $\sum_{a \in A} am_a = 0$ in ΛeM. Now, let $x \in I$. By Lemma **??**.(5), write $x = \sum a'g'b'$ with $a' \in A, b' \in B$, and $g' \in K[G]$. For each term $a \otimes m_a$ of α, if $b'a \in K[G]$, then $a'g'b'a \otimes m_a = a' \otimes g'b'am_a$, and if $b'a = 0$, then $a'g'b'a \otimes m_a = 0$. Hence $a'g'b'\alpha = \sum_{a \in A} a' \otimes g'b'am_a = a' \otimes g'b'(\sum_{a \in A} am_a) = 0$. Therefore $x\alpha = 0$, as desired. The last part of the statement of the lemma is obvious by using x equal to the left identity of I. ∎

It follows from the lemma that for any left $K_0[S]$-module both modules M^* and M^{**} are left $K_0[S/I]$-modules.

We also mention the following well-known lemma on change of rings (see Proposition 7.2.2 in [?]).

LEMMA 2.3. *Let R, S be rings with identity. If $R \to S$ is a ring homomorphism, then for any left S-module M,*

$$pd_R(M) \le pd_S(M) + pd_R(S).$$

In order to show the main theorem of this section, we also need the following lemma.

LEMMA 2.4. *Assume S is a monoid with an ideal U that is isomorphic to a non-null Rees matrix semigroup $\mathcal{M}^0(G; m, n; P)$ and $S \neq U$. Let $T = S/U$. Consider $K_0[T]$ as a left $K_0[S]$-module, then $pd_{K_0[S]}(K_0[T]) \le 1$. Furthermore, $K_0[T]$ is projective if and only if $K_0[U]$ has a right identity.*

Proof. Obviously, we have a short exact sequence

$$0 \to K_0[U] \to K_0[S] \to K_0[T] \to 0.$$

By Lemma **??**, $K_0[U]$ is a projective $K_0[S]$-module and thus $pd_{K_0[S]}(K_0[T]) \leq 1$. Furthermore, $K_0[T]$ is projective if and only if the sequence splits, or equivalently if $K_0[U]$ has a right identity. ∎

THEOREM 2.5. *Let S be a monoid, let U be an ideal which is isomorphic to a non-null Rees matrix semigroup $\mathscr{M}^0(G; m, n; P)$, and let $T = S/U$. Then, for any ring K with identity,*

$$\text{l.gl.dim}(K_0[S]) \leq \max\{\text{l.gl.dim}(K[G]), \text{l.gl.dim}(K_0[T]) + \sigma(U)\},$$

where

$$\sigma(U) = \begin{cases} 0, & \textit{if } K_0[U] \textit{ has an identity} \\ 1, & \textit{if } K_0[U] \textit{ has a left or right identity, but not an identity} \\ 2, & \textit{if } K_0[U] \textit{ does not have a left nor a right identity.} \end{cases}$$

Proof. As before, denote $\Lambda = K_0[S]$. From the basic exact sequences defined above, for any left Λ-module M, we have (Exercise 14, p. 463 in [**?**]):

$$pd_\Lambda(M) \leq \max\{pd_\Lambda(\Lambda e M),\ pd_\Lambda(M^*)\},$$

and by 7.1.6 in [**?**],

$$pd_\Lambda(\Lambda e M) \leq \max\{pd_\Lambda(\Lambda e \otimes_{K[G]} eM),\ pd_\Lambda(M^{**}) + 1\}.$$

By Lemma **??**, $\Lambda e = Ie$ is a free right $K[G]$-module, hence $pd_\Lambda(\Lambda e \otimes_{K[G]} eM) \leq pd_{K[G]}(eM)$.

Thus the second inequality becomes

$$pd_\Lambda(\Lambda e M) \leq \max\{pd_{K[G]}(eM),\ pd_\Lambda(M^{**}) + 1\}.$$

By Lemmas **??** and **??**, we have

$$\begin{aligned} pd_\Lambda(M^*) &\leq pd_{K_0[T]}(M^*) + pd_\Lambda(K_0[T]) \\ &\leq \begin{cases} \text{l.gl.dim}(K_0[T]), & \text{if } K_0[U] \text{ has a right identity} \\ \text{l.gl.dim}(K_0[T]) + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

If $K_0[U]$ does not have a left identity, then again by Lemmas **??** and **??**, we have

$$\begin{aligned} pd_\Lambda(M^{**}) &\leq pd_{K_0[T]}(M^{**}) + pd_\Lambda(K_0[T]) \\ &\leq \begin{cases} \text{l.gl.dim}(K_0[T]), & \text{if } K_0[U] \text{ has a right identity} \\ \text{l.gl.dim}(K_0[T]) + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} pd_\Lambda(M) &\leq \max\{pd_\Lambda(\Lambda eM), pd_\Lambda(M^*)\} \\ &\leq \max\{pd_{K[G]}(eM), pd_\Lambda(M^{**})+1, pd_\Lambda(M^*)\} \\ &\leq \max\{\text{l.gl.dim}(K[G]), \text{l.gl.dim}(K_0[T]) + \sigma(U)\}. \end{aligned}$$

So the result follows in this case.

On the other hand, if $K_0[U]$ has a left identity, then, by Lemma **??**, $M^{**} = 0$ for any left Λ-module M. From the second basic exact sequence, we have using Lemma **??**,

$$\begin{aligned} pd_\Lambda(\Lambda eM) &= pd_\Lambda(\Lambda e \otimes_{K[G]} eM) \\ &\leq pd_{K[G]}(eM) \\ &\leq \text{l.gl.dim}(K[G]). \end{aligned}$$

Hence

$$\begin{aligned} pd_\Lambda(M) &\leq \max\{pd_\Lambda(\Lambda eM), pd_\Lambda(M^*)\} \\ &\leq \max\{\text{l.gl.dim}(K[G]), \text{l.gl.dim}(K_0[T]) + \sigma(U)\}. \end{aligned}$$

So the result follows. ∎

Theorem **??** also allows us to find the following upper bound for the left global dimension of contracted semigroup algebras $K_0[S]$ of more general semigroups S.

THEOREM 2.6. *Let S be a monoid with a sequence of ideals $S = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1}$, where $I_{n+1} = \{\theta\}$ or $\emptyset$. Let K be a ring with an identity. Assume that, for all $1 \leq i \leq n$, I_i/I_{i+1} is a non-null Rees matrix semigroup $\mathcal{M}^0(G_i; m_i, n_i; P_i)$. Let σ be defined as in Theorem* **??** *and let $\mu_j(S) = \sigma(I_j/I_{j+1}) + \cdots + \sigma(I_n/I_{n+1})$, for $1 \leq j \leq n$. Let $\mu_{n+1}(S) = 0$. Then*

$$\text{l.gl.dim}\, K_0[S] \leq \max\{\text{l.gl.dim}(K[G_j]) + \mu_{j+1}(S) : j = 1, \ldots, n\}.$$

Proof. If $n = 1$, then $K_0[S] = K[G]$, a group algebra. Hence, in this situation the assertion is obvious. We now prove the result by induction on n. For $n \geq 2$ consider the factor semigroup S/I_n. By Theorem **??**, we have

$$\text{l.gl.dim}\, K_0[S] \leq \max\{\text{l.gl.dim}(K[G_n]), \text{l.gl.dim}(K_0[S/I_n]) + \sigma(I_n)\}.$$

By the induction hypothesis, we have

$$\text{l.gl.dim}\, K_0[S/I_n] \leq \max\{\text{l.gl.dim}(K[G_j]) + \mu_{j+1}(S/I_n) : j = 1, \cdots, n-1\}.$$

As $\mu_{j+1}(S/I_n) = \sigma(I_{j+1}/I_{j+2}) + \cdots + \sigma(I_{n-1}/I_n)$, we therefore get

$$\begin{aligned} \text{l.gl.dim}\, K_0[S] \leq & \max\{\text{l.gl.dim}(K[G_n]), \text{l.gl.dim}(K[G_j]) + \mu_{j+1}(S/I_n) \\ & \qquad + \sigma(I_n) : j = 1, \ldots, n-1\} \\ & \leq \max\{\text{l.gl.dim}(K[G_j]) + \mu_{j+1}(S) : j = 1, \ldots, n\}. \end{aligned}$$

∎

As an application of Theorem **??**, we obtain the exact value of the global dimension of $K_0[S^1]$ for non-null Rees matrix semigroups S.

THEOREM 2.7. *Let S be a non-null Rees matrix semigroup* $\mathcal{M}^0(G; n, m; P)$ *with G a finite group. If K is a field of characteristic not dividing the order of G, then* $\text{l.gl.dim}(K_0[S^1]) = \sigma(S)$.

Proof. The assumption implies that $\text{l.gl.dim}(K[G]) = 0$. From Theorem **??**, we have $\text{l.gl.dim}(K_0[S^1]) \leq 1$ provided that $\sigma(S) \leq 1$. It is obvious that $\text{l.gl.dim}(K_0[S^1]) = 0$ if and only if $\mu_1(S^1) = \sigma(S) = 0$. Hence the theorem holds for $\sigma(S) \leq 1$.

Next assume $\sigma(S) = 2$. We may assume $\text{l.gl.dim}(K_0[S^1]) \geq 1$. Hence by 7.1.8 of [**?**], we have

$$\text{l.gl.dim}(K_0[S^1]) = 1 + \sup\{pd_{K_0[S^1]}(I) : I \subseteq K_0[S^1] \text{ is a left ideal}\}.$$

So to prove the theorem, it is sufficient to find a left ideal of projective dimension 1. Since $K[G]$ is semisimple, say $K[G] = M_{k_1}(D_1) \oplus \cdots \oplus M_{k_r}(D_r)$ for some division rings D_i, we can decompose $K_0[S]$ naturally as the sum of $\mathcal{M}(M_{k_i}(D_i); n_i, m_i; P_i)$ for $1 \leq i \leq r$. Here $P = P_1 \oplus \cdots \oplus P_r$ and entries of P_i belong to $M_{k_i}(D_i)$ for all $1 \leq i \leq r$. Since $K_0[S]$ does not have a left identity, there exists i_0 such that $\mathcal{M}(M_{k_{i_0}}(D_{i_0}); n_{i_0}, m_{i_0}; P_{i_0}) \cong \mathcal{M}(D_{i_0}; k_{i_0}n_{i_0}, k_{i_0}m_{i_0}; \bar{P}_{i_0})$ does not have a left identity. Here $\bar{P}_{i_0}$ denotes the $k_{i_0}m_{i_0} \times k_{i_0}n_{i_0}$ matrix obtained from P_{i_0} by erasing the matrix brackets of all the entries. Hence $\bar{P}_{i_0}$ does not have a left inverse. From Corollary 2.3 on p. 439 in [**?**], we have $\text{ann}_r(\mathcal{M}(M_{n_{i_0}}(D_{i_0}); n_{i_0}, m_{i_0}; P_{i_0})) \neq 0$ and thus $\text{ann}_r(K_0[S]) \neq 0$. (By $\text{ann}_r(\ )$ we denote the right annihilator.) Let $0 \neq \delta \in \text{ann}_r(K_0[S])$ and let $I = K_0[S^1]\delta$. Clearly $K_0[S^1]\delta \cong K$ as left $K_0[S^1]$ modules. By **??** and the fact that $K_0[S]$ does not have a right identity, $pd_{K_0[S^1]}(K) = 1$. The result follows. ∎

THEOREM 2.8. *Let S be a non-null Rees matrix semigroup* $\mathcal{M}^0(G; n, m; P)$ *with G a finite group. Let K be a field. Then the following are equivalent*:

(1) $K_0[S^1]$ *is hereditary*;

(2) $K[G]$ *is semisimple and* $K_0[S]$ *has a left or right identity*;

(3) $K[G]$ *is semisimple and there exists* $a \in K_0[S]$ *that is not a right or not a left divisor of zero in* $K_0[S]$;

(4) $K[G]$ *is semisimple and* $\operatorname{ann}_r(K_0[S]) = 0$ *or* $\operatorname{ann}_l(K_0[S]) = 0$.

Proof. $1 \Rightarrow 2$. Since $K[S^1]$ is hereditary, $K[G] \cong eK_0[S^1]e$ is also hereditary by Proposition 7.8.9 in [**?**]. Since G is finite, Corollary 10.3.7 in [**?**] implies that $K[G]$ is semisimple. So (2) follows from Theorem **??**.

$2 \Rightarrow 3$ and $3 \Rightarrow 4$ are clear.

$4 \Rightarrow 2$. This is shown in the last part of the proof of Theorem **??**. ∎

Remark. If G is trivial, then the above conditions are equivalent to $\operatorname{rank}(P) = \min\{m, n\}$. In general, the above conditions are equivalent to $\operatorname{rank}(P_i) = k_i \cdot \min\{m_i, n_i\}$ for all $1 \leq i \leq r$ when $K[G] = M_{k_1}(D_1) \oplus \cdots \oplus M_{k_r}(D_r)$ and $K[S] = \bigoplus_{i=1}^{r} \mathcal{M}(D_i; k_i n_i, k_i m_i; P_i)$. Here $\operatorname{rank}(P_i)$ is defined as the dimension of the column space of P_i, see [**?**].

3. NILPOTENT SEMIGROUPS

Recall that a Dedekind prime ring is a ring which is a hereditary Noetherian prime ring and a maximal order. Also, if R is a hereditary Noetherian prime ring satisfying a polynomial identity, then R is obtained from a Dedekind prime ring by a finite iteration of the process of forming idealizers of generative isomaximal right ideals. In fact, R is Morita equivalent to a Dedekind prime ring (Theorem 13.7.15, 5.6.12 and 5.6.8, [**?**]). In this section, we prove that if S is a nilpotent semigroup and $K[S]$ is an HNP, even without the PI condition, then $K[S]$ is a Dedekind prime ring and thus $K[S]$ is a maximal order. Throughout this section we assume that every (contracted) semigroup algebra $K[S]$ has an identity element.

First we recall the definition of a nilpotent semigroup in the sense of Malcev [**?**]. Let $x, y \in S$ and let $w_1, w_2, \ldots$ be elements of S^1 (as in [**?**], we allow $w_i = 1$; so this definition is slightly stronger than the one given by Malcev). Consider the sequence (x_n) defined inductively as follows:

$$x_0 = x, \qquad y_0 = y,$$

and for $n \geq 0$,

$$x_{n+1} = x_n w_{n+1} y_n, \qquad y_{n+1} = y_n w_{n+1} x_n.$$

We say the identity $X_n = Y_n$ is satisfied in S if $x_n = y_n$ for all $x, y \in S$ and $w_1, w_2, \ldots \in S^1$. A semigroup S is said to be nilpotent of class n if S satisfies the identity $X_n = Y_n$ and n is the least positive integer with this property.

Recall that if S is a group, this definition coincides with the classical notion of nilpotency. The following Proposition is crucial in the proof of the results in this section. Note that if P is an ideal of $K[S]$ then $\sim_P$ denotes the congruence on S defined by $\sim_P = \{(s,t) \mid s, t \in S, s - t \in P\}$. Clearly there exist natural $K[S]$-epimorphisms, $K[S] \to K[S/\sim_P] \to K[S]/P$. And one can identify the semigroup $S/\sim_P$ with its image in $K[S]/P$ (see Lemma 4.5 in [**?**]).

PROPOSITION 3.1 [**?**, Theorem 3.5]. *Let S be a nilpotent semigroup, K a field and P a prime ideal of $K[S]$ such that $K[S]/P$ is left Goldie with classical ring of quotients $M_n(D)$ and D a division ring. Then the semigroup $S/\sim_P$ has an ideal chain*

$$S/\sim_P \; = I_r \supset I_{r-1} \supset \cdots \supset I_1 = I \supset I_0,$$

where $I_0 = \theta$ if S has a zero element and $I_0 = \emptyset$ otherwise, and for all $j > 0$, I_j consists of matrices in $S/\sim_P \; \subset M_n(D)$ of rank less than or equal to some positive integer n_j (in particular, I is the ideal of matrices with the minimal nonzero rank in $M_n(D)$), such that

(1) *I is uniform (in the sense of Okninski in [**?**]) in a completely 0-simple inverse subsemigroup $\hat{I}$ of $M_n(D)$ with finitely many idempotents and $\hat{S} = (S/\sim_P) \cup \hat{I}$ is a nilpotent subsemigroup of $M_n(D)$, and $\hat{I}$ is an ideal of $\hat{S}$.*

(2) *$K\{I\} \subseteq K[S]/P \subseteq K\{\hat{I}\}$, where $K\{\hat{I}\}$ denotes the subalgebra of $M_n(D)$ generated by $\hat{I}$; moreover $M_n(D)$ is the common classical ring of quotients of these three algebras and $K\{\hat{I}\}$ is a left and right localization of $K\{I\}$ with respect to an Ore set.*

(3) *Denote by G a maximal subgroup of $\hat{I}$. There exists a prime ideal Q of $K[G]$ such that $K[G]/Q$ is a Goldie ring and $K\{\hat{I}\} \cong M_q(K[G]/Q)$, where q is the number of nonzero idempotents of $\hat{I}$; moreover G is the group of quotients of $I \cap G$.*

If, in the previous proposition, $P = 0$, then one obtains more information.

LEMMA 3.2 [**?**, Lemma 1.6]. *Let S be a nilpotent semigroup, K a field, $P = K \cdot \theta$ if S has a zero element, otherwise $P = 0$. If $K_0[S] = K[S]/P$ is a prime left Goldie ring satisfying the ascending chain condition on two-sided ideals, then, with notations as in Proposition* **??**, *$Q = 0$, $K\{\hat{S}\} = K\{\hat{I}\} = K_0[\hat{I}]$, G is poly-infinite cyclic and $q = n$.*

We now can show that an hereditary Noetherian prime semigroup algebra of a nilpotent semigroup is a maximal order.

PROPOSITION 3.3. *Let S be a nilpotent semigroup and K a field. If $K_0[S]$ is hereditary prime left Goldie ring satisfying ascending chain condition on* *two-sided ideals, then G is infinite cyclic or trivial, and $K_0[S]$ is a maximal order.*

Proof. We use the same notation as in Lemma **??**. Note here that $K_0[\hat{I}] \cong M_q(K[G])$ is a localization of $K_0[I]$ with respect to an Ore set C of regular elements (regular in $M_q(K[G])$). First we show that $K_0[\hat{I}] \cong M_q(K[G])$ is also a localization of $K_0[S]$ with respect to the Ore set C. Since elements of C are regular, it is sufficient to show that C satisfies the Ore condition in $K_0[S]$. Let $s \in K_0[S]$ and $c \in C$. Then $sc^{-1} \in K_0[\hat{I}] = C^{-1}K_0[I]$, so $sc^{-1} = d^{-1}r$ for some $d \in C$ and $r \in K_0[I]$. Hence $ds = rc$ and thus $Cs \cap K_0[S]c \neq \emptyset$.

As a localization of an hereditary algebra, $K_0[\hat{I}] \cong M_q(K[G])$ is hereditary, and thus $K[G]$ is a hereditary algebra. By the results of Goursaud and Valette [**?**] (or Theorem 17.5 in [**?**]), G is either finite-by-(infinite cyclic), and the order of the torsion subgroup of G is invertible in K, or G is locally finite and countable, and the order of every element of G is invertible in K. Since $K\{\hat{I}\} \cong M_q(K[G])$ is prime, and thus also $K[G]$, and because G is nilpotent, it is well known that G is torsion-free. Hence G is infinite cyclic or trivial.

Now, we show that $K_0[S]$ is a maximal order when G is infinite cyclic. For this it is sufficient to show that $K_0[S]$ is a Dedekind prime ring. Because of Proposition 5.6.3 in [**?**] we only have to show that any idempotent ideal of $K_0[S]$ is trivial. So suppose J is a nontrivial idempotent ideal of $K_0[S]$. Then since $K_0[\hat{I}]$ is Noetherian and a localization of $K_0[S]$, $JK_0[\hat{I}]$ is an idempotent ideal of $K_0[\hat{I}]$ (Theorem 1.31 in [**?**]). Since $K_0[\hat{I}]$ is a prime left principal ideal ring by the result of Jespers and Wauters [**?**, Theorem 1.1] and because prime principal ideal ring do not contain nontrivial idempotent ideals, we obtain a contradiction. ∎

As a consequence we obtain the following structure theorem.

THEOREM 3.4. *Let S be a nilpotent semigroup. The following conditions are equivalent*:

(1) $K_0[S]$ *is a HNP*;

(2) $K_0[S]$ *is a prime Asano-order*;

(3) $K_0[S]$ *is a prime left principal ideal ring*;

(4) $S \cong \mathscr{M}^0(e; n, n; \Delta)$ *or* $S \cong \mathscr{M}^0(\{x^i \mid i \in N\}; n, n; \Delta)$ *or* $S \cong \mathscr{M}^0(\{x^i \mid i \in Z\}; n, n; \Delta)$ *and thus* $K_0[S] \cong M_n(K)$ *or* $K_0[S] \cong M_n(K[X])$ *or* $K_0[S] \cong M_n(K[X, X^{-1}])$.

Proof. Note that because of Theorem 1.5 in [**?**], the last three conditions are equivalent. It remains to show that (1) and (4) are equivalent. So assume $K_0[S]$ is hereditary Noetherian and prime. Then by Proposition **??**

and Proposition **??**, and the fact that $\hat{I}$ is completely 0-simple inverse, $K_0[S]$ is a maximal order and $\hat{I} = \mathscr{M}^0(G; n, n; \Delta)$ with G trivial or the infinite cyclic group. As pointed out on p. 5063 in [**?**], these conditions are sufficient to prove (4). That (4) implies (1) is obvious. ∎

4. CANCELLATIVE SEMIGROUPS

Let S be a cancellative monoid. Assume $K[S]$ is hereditary and Noetherian. Then S has a group G of fractions by Proposition 7.12 in [**?**]. So $K[G]$ is a localization of $K[S]$ and $K[G]$ is also hereditary and Noetherian. Such group algebras have been discussed by Dicks [6, 7].

THEOREM 4.1 (Dicks [**?**, **?**]) (see also Theorem 17.4, in [**?**]). *A group algebra $K[G]$ is hereditary if and only if*

(*) *G is the fundamental group of a connected graph of finite groups with invertible orders in K.*

Moreover, if G is finitely generated, then the above is equivalent to any of the following conditions:

(1) *G has a free subgroup of finite index, and the orders of finite subgroups of* G *are invertible in K*;

(2) *G is the fundamental group of a finite connected graph of finite groups of orders invertible in K.*

It is well known when a fundamental group G of a connected graph of finite groups has no free subgroup of rank 2. By $G_1 * G_2$ we denote the free product of the groups G_1 and G_2. The cyclic group of order two is denoted by Z_2.

LEMMA 4.2 ([**?**]). *A fundamental group* G *of a connected graph of finite groups has no free subgroup of rank* 2 *if and only if either of the following conditions holds*:

(1) *G is countable locally finite*;

(2) *G is finite-by-*(*infinite cyclic*);

(3) *G is finite-by-$Z_2 * Z_2$.*

Since $K[G]$ is Noetherian, the group G satisfies the ascending chain condition on subgroups. So from Theorem 4.1 and Lemma **??** we obtain that G is either finite or finite-by-(infinite cyclic) or finite-by-$Z_2 * Z_2$, and moreover, the orders of finite subgroups of G are invertible in K. In the first case, we get that $K[G]$ and thus $K[S]$ is semisimple Artinian.

Now we discuss the second case, that is, G is finite-by-(infinite cyclic). We will prove $K[S]$ is a principal left ideal ring. First note that by Passman's result in [?] (see also Theorem 1.1 in [?]) $K[G]$ is a principal (left and right) ideal ring.

We note that $K[G]$ is semiprime because all finite subgroups of G have invertible order in K. Hence by Theorem 19 ([?], Chap. 7), the semigroup algebra $K[S]$ is also semiprime.

Now we claim $K[S]$ is a maximal order. Since $K[S]$ is a semiprime Noetherian hereditary ring, the semigroup algebra $K[S]$ can be decomposed into a finite direct sum of hereditary Noetherian prime rings:

$$K[S] = \bigoplus_{i=1}^{n} e_i K[S], n \geq 1,$$

where each e_i is a primitive central idempotent. Hence to prove the claim it is sufficient to show that each $e_iK[S]$ is a Dedekind prime ring, and thus we only need to show that each $e_iK[S]$ has no nontrivial idempotent ideal. So suppose I is an idempotent ideal of $e_iK[S]$. Since $K[G]$ is a Noetherian ring and a localization of $K[S]$, it follows that $e_iK[G]$ is also a Noetherian ring and a localization of $e_iK[S]$. Hence $Ie_iK[G]$ is a two-sided idempotent ideal of $e_iK[G]$. But the latter is prime principal ideal ring and thus $Ie_iK[G] = 0$ or $Ie_iK[G] = e_iK[G]$, as required.

Finally, we prove $K[S]$ is a left principal ideal ring. Let H be a finite normal subgroup of G and $g \in G$ so that $G/H = \langle gH \rangle$ is an infinite cyclic group. Then $K[G] = K[H] * (G/H) = K[H][g, g^{-1}; \sigma]$, a skew Laurent polynomial ring over $K[H]$. Obviously, $K[S] \subseteq K[H][g, g^{-1}, \sigma]$ and $S \subseteq G = \langle g, H \rangle$. Let $A = \{i \in Z : S \cap Hg^i \neq \emptyset\}$. Clearly A is a nontrivial subsemigroup of Z. If A is a group, then $A = mZ$ for some $m \geq 1$. Hence $S \subseteq \bigcup_{i \in Z} Hg^{im} \subseteq G$. But $G = \langle S, S^{-1} \rangle$ implies $\bigcup_{i \in Z} Hg^{im} = G$ and thus $m = 1$. It follows that $S = G$, a finite-by-infinite cyclic group.

Assume now A is not a group. Then, without loss of generality, we may asssume $A \subseteq \mathbf{N} \cup \{0\}$. Since submonoids of $\mathbf{N} \cup \{0\}$ are well known (see for example Theorem 2.4 in [?]), there exists K_0 such that $k \in A$ for all $k \geq K_0$. Denote $H_k = \{h \in H \mid hg^k \in S\}$. Hence $H_k \neq \emptyset$ for all $k \geq K_0$. Because H is finite, the automorphism σ has finite order, say α. Let $j = \alpha \cdot |H| \cdot K_0$, then $g^j \in S$ and thus $1 \in H_j$. Obviously, $H_j \subseteq H_{2j} \subseteq \cdots \subseteq H_{nj} \subseteq \cdots$. Since H is a finite group, there exists a multiple j_0 of j such that $H_{mj_0} = H_{j_0}$ for any $m \geq 1$. So H_{j_0} is a subgroup since it is multiplicatively closed. Clearly also

$$H_{j_0+1} \subseteq H_{2j_0+1} \subseteq \cdots \subseteq H_{nj_0+1} \subseteq \cdots$$

$$\cdots$$

$$H_{2j_0-1} \subseteq H_{3j_0-1} \subseteq \cdots \subseteq H_{(n+1)j_0-1} \subseteq \cdots.$$

Hence as H is finite, then there exists a multiple v of j_0, such that

$$\begin{aligned} H_v &= H_{2v} = \cdots = H_{nv} = \cdots \\ H_{v+1} &= H_{2v+1} = \cdots = H_{nv+1} = \cdots \\ &\cdots \\ H_{2v-1} &= H_{3v-1} = \cdots = H_{(n+1)v-1} = \cdots . \end{aligned}$$

We claim that $H_v = H$. Let $h \in H \subseteq G = SS^{-1}$. Then $h = p^{-1}t$ for some $p, t \in S$. So $h = p^{-v}(p^{v-1}t)$. Replacing p by p^v we may assume $p = h_{vk}g^{vk}$ for some $h_{vk} \in H_{vk}$ and some positive number k. Hence $t = h'_{vk}g^{vk}$ for some $h'_{vk} \in H_{vk}$.

As g^{vk} acts trivially on H, we get

$$\begin{aligned} h &= (h_{vk}g^{vk})^{-1}(h'_{vk}g^{vk}) \\ &= g^{-vk}h_{vk}^{-1}h'_{vk}g^{vk} \\ &= h_{vk}^{-1}h'_{vk} \in H_{vk}. \end{aligned}$$

Hence there exists a positive integer k such that $H = H_{vk} = H_v$. Since $H_{v+i} \neq \emptyset$, there exists $h_0 \in H_{v+i}$ such that $h_0 g^{v+i} \in S$ for $1 \leq i \leq v$. Hence $g^{2v+i} = h_0^{-1}g^v \cdot h_0 g^{v+i} \in S$ (again we use that v is a multiple of $\alpha \cdot |H|$). Thus $g^l \in S$ for all $l \geq 2v$ and $H_t = H$ for all $t \geq 3v$. Consider the ideal $I = \bigoplus_{t \geq 3v} K[H]g^t$ of $K[S]$. It follows that $HI \subseteq I$ and $gI \subseteq I$. Since $K[S]$ is a maximal order we obtain that $g \in S$ and $H \subseteq S$. Therefore, $S = \bigcup_{i \in N} Hg^i$ and $K[S] = \bigoplus_{i \in N} K[H]g^i \cong K[H][g, \sigma]$.

We now show that the central idempotents in $K[H]$ remain central in $K[S]$. (Our proof is similar to that of Lemma 2.3 in [**?**]). Write $K[H] = A_1 \oplus \cdots \oplus A_n$, where each A_i is simple Artinian with unit element, say e_i. It is sufficient to prove that each e_i is central in $K[S]$. We do this for $i = 1$. Since conjugation by σ permutes the idempotents $e_1, \ldots, e_n$, we get $g^{-1}A_1 = A_m g^{-1}$, for some $1 \leq m \leq n$. We need to show that $m = 1$. Suppose the contrary. Then consider the left ideal $L = A_1 + K[S]g$ of $K[S]$. Calculating in $K[G]$ we get

$$\begin{aligned} (e_1 g^{-1}) \cdot L &= e_1 g^{-1}(A_1 + K[S]g) \\ &= e_1 g^{-1} A_1 + e_1 g^{-1} K[S]g \\ &= e_1 A_m g^{-1} + e_1 K[S] \subseteq L, \end{aligned}$$

because $e_1 A_m = 0$ and $e_1 K[H] \subseteq A_1$. Since $K[S]$ is a maximal order, it follows $e_1 g^{-1} \in K[S]$, a contradiction. Hence the condition (c) in the following theorem is satisfied.

Now we discuss the third case, that is, G contains a finite p'-subgroup H and $G/H \cong \langle a, b \mid bab = a^{-1}, b^2 = 1 \rangle$; where p is the characteristic of the field K. We can express any element of G as hx^iy or $\bar{h}x^j$ where $h, \bar{h} \in H$, $i, j \in Z$, and x, y are pre-images in G of a and b, respectively. Because G is the group of quotients of S, there must exist an element in S with form hx^iy with $h \in H$, $i \in Z$. Consider the abelian subgroup $N = \langle x \rangle$ of G. Because N has finite index in G and $G = SS^{-1}$, we get $N = (S \cap N)(S \cap N)^{-1}$ by Lemma 7.5 in [?]. We now claim that, if $x^t \in S$ for some positive integer t, then $x^{-kt} \in S$ for some $k \geq 1$. Indeed, since $K[S]$ is Noetherian, by Lemma 1.3 in [?], for any $c, d \in S$, there exists a positive integer r such $c^r d \in dS$. We apply this to $c = x^t$ and $d = hx^iy$. Then $c^r d = x^{rt}hx^iy = hx^iyg$ for some $g \in S$. Hence it is easily seen that there exists $h' \in H$ such that $g = yx^{rt}yh' = x^{-rt}h'$ and thus $x^{-kt} \in S$ for some positive integer k. This proves the claim. It follows that $S \cap N$ is a subgroup of N. Hence $N = (S \cap N)(S \cap N)^{-1} = S \cap N$. So $N \subseteq S$ and thus $S = G$ is a finite p'-by-$Z_2 * Z_2$ group.

THEOREM 4.3. *Let S be a cancellative monoid and K a field of characteristic p (not necessarily nonzero). Then the following are equivalent:*

(1) *The semigroup algebra $K[S]$ is a Noetherian hereditary ring.*

(2) *The semigroup S satisfies one of the following conditions:*

(a) *S is a finite p'-group;*

(b) *S is a finite p'-by-infinite cyclic group;*

(c) *S contains a finite p'-subgroup H and a non-periodic element x such that $S = \bigcup_{i \in N} Hx^i$, $xH = Hx$, and every central idempotent of $K[H]$ remains central in $K[S]$;*

(d) *S is a finite p'-by-$Z_2 * Z_2$ group.*

Proof. That (1) implies (2) follows from the previous discussion.

Conversely, if S satisfies one of the conditions (a), (b), the result is obvious. If S satisfies case (c), then $K[S]$ is a skew polynomial ring $K[H][g, \sigma]$ with $\text{l.gl.dim}\, K[S] = \text{l.gl.dim}\, K[H] + 1$ (see Theorem 7.5.3 in [?]). Thus $K[S]$ is hereditary. If S satisfies case (d), then (1) follows from the result of Dicks. ∎

Note also, by Theorem 2.1 of [?] and Theorem 2.13 in [?], the semigroup algebra $K[S]$ is a semiprime principal left ideal ring if and only if one of the conditions (a), (b), (c) hold. However, (d) does not give a principal left ideal ring. Indeed, it is well known (see for example in [?]) that the group algebra of the infinite dihedral group $Z_2 * Z_2$ is not a maximal order.

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